代数幾何学シンポジウム記録
2008年度 pp.85-89

POLARIZED ENDOMORPHISMS ON NORMAL PROJECTIVE VARIETIES

DE-QI ZHANG

ABSTRACT. This is the summary of the paper [14]. We show that polarized endomorphisms of rationally connected threefolds with at worst terminal singularities are equivariantly built up from those on $\mathbb{Q}$-Fano threefolds, Gorenstein log del Pezzo surfaces and $\mathbb{P}^1$. Similar results are obtained for polarized endomorphisms of uniruled threefolds and fourfolds. As a consequence, we show conceptually that every smooth Fano threefold with a polarized endomorphism of degree > 1, is rational.

1. INTRODUCTION

We work over the field $\mathbb{C}$ of complex numbers. We study *polarized* endomorphisms $f : X \to X$ of varieties X, i.e., those f with $f^*H \sim qH$ for some $q > 0$ and some ample line bundle H. Every surjective endomorphism of a projective variety of Picard number one, is polarized. If $f = [F_0 : F_1 : \cdots : F_n] : \mathbb{P}^n \to \mathbb{P}^n$ is a surjective morphism and $X \subset \mathbb{P}^n$ a f-stable subvariety, then $f^*H \sim qH$ and hence $f|X : X \to X$ is polarized; here $H \subset X$ is a hyperplane and $q = \deg(F_i)$. If A is an abelian variety and $m_A : A \to A$ the multiplication map by an integer $m \neq 0$, then $m_A^*H \sim m^2H$ and hence m_A is polarized; here $H = L + (-1)^*L$ with L an ample divisor, or H is any ample divisor with $(-1)^*H \sim H$. One can also construct polarized endomorphisms on quotients of $\mathbb{P}^n$ or A. So there are many examples of polarized endomorphisms f. See [16] for the many conjectures on such f.

In [11], it is proved that a normal variety X with a non-isomorphic polarized endomorphism f either has only canonical singularities with $K_X \sim_{\mathbb{Q}} 0$ (and further is a quotient of an abelian variety when $\dim X \leq 3$), or is uniruled so that f descends to a polarized endomorphism f_Y of the non-uniruled base variety Y (so $K_Y \sim_{\mathbb{Q}} 0$) of a specially chosen maximal rationally connected fibration $X \dashrightarrow Y$. By the induction on dimension and since Y has a dense set of f_Y-periodic points $y_0, y_1, \ldots$ (cf. [2, Theorem 5.1]), the study of polarized endomorphisms is then reduced to that of rationally connected varieties Γ_{y_i} as fibres of the graph $\Gamma = \Gamma(X/Y)$ (cf. [11, Remark 4.3]).

The study of non-isomorphic endomorphisms of singular varieties (like Γ_{y_i} above) is very important from the dynamics point of view, but is very hard

1991 *Mathematics Subject Classification.* 14E20, 14J45, 14E08, 32H50.
Key words and phrases. polarized endomorphism, uniruled variety, rationality of variety.

1

even in dimension two and especially for rational surfaces; see [9] (about 150 pages).

We consider polarized endomorphisms of rationally connected varieties (or more generally of uniruled varieties) of dimension ≥ 3. Theorem 1.1 – 1.8 below are our main results.

Theorem 1.1. *Let X be a $\mathbb{Q}$-factorial threefold having only terminal singularities and a polarized endomorphism of degree $q^3 > 1$. Suppose that X is rationally connected. Then we have :*

1. *There is an $s > 0$ such that $(f^s)^*{}_{|N^1(X)} = q^s \,\mathrm{id}$.*
2. *Either X is rational, or $-K_X$ is big.*
3. *There are only finitely many irreducible divisors $M_i \subset X$ with the Iitaka D-dimension $\kappa(X, M_i) = 0$.*

Theorem 1.1 (3) apparently does not hold on an abelian variety A with a subtorus of codimension one, though the multiplication map m_A is polarized as mentioned above. Neither it holds for $X = S \times \mathbb{P}^1$, where S is a rational surface with infinitely many (-1)-curves (the blowup of nine general points of $\mathbb{P}^2$ is such S as observed by Nagata).

Theorem 1.1 (1) above strengthens (in our situation) Serre's result [12] on a conjecture of Weil (in the projective case): **(Serre)** If f is a polarized endomorphism of degree $q^{\dim X} > 1$ of a smooth variety X then every eigenvalue of $f^*|N^1(X)$ has the same modulus q.

The proof of Theorem 1.2 below is conceptually done. In a recent paper [15], we have removed the polarizedness assumption in Theorem 1.2.

Theorem 1.2. *Let X be a smooth Fano threefold with a polarized endomorphism of degree > 1. Then X is rational.*

A klt $\mathbb{Q}$-Fano variety has only finitely many extremal rays. A similar phenomenon occurs in the quasi-polarized case.

Theorem 1.3. *Let X be a $\mathbb{Q}$-factorial rationally connected threefold having only Gorenstein terminal singularities and a quasi-polarized endomorphism of degree > 1. Then X has only finitely many K_X-negative extremal rays.*

We expect a possible application of Theorem 1.4 below (see Theorem 1.7 for a more detailed version) to the Dynamic Manin-Mumford conjecture for (X, f) formulated by S. -W. Zhang in [16, Conjecture 1.2.1]. This conjecture for (X, f) is essentially equivalent to that for (X_r, g_r) because f^{-1}, as seen in Theorem 1.7, preserves the maximal subset of X where the birational map $X \dashrightarrow X_r$ is not holomorphic.

Further, X_r is better to be dealt with because it has a fibration structure preserved by g_r. *The existence of such a fibration $\pi : X_r \to Y$ is guaranteed* when X is uniruled by the recent development in MMP.

Theorem 1.4. *Let X be a $\mathbb{Q}$-factorial n-fold, with $n \in \{3, 4\}$, having only log terminal singularities and a polarized endomorphism f of degree $q^n > 1$.*

Let $X = X_0 \dashrightarrow X_1 \cdots \dashrightarrow X_r$ be a composition of divisorial contractions and flips. Replacing f by its positive power, we have:

(1) *The dominant rational maps $g_i : X_i \dashrightarrow X_i$ $(0 \le i \le r)$ (with $g_0 = f$) induced from f, are all holomorphic.*
(2) *Let $\pi : X_r \to Y$ be an extremal contraction with $\dim Y \le 2$. Then g_r is polarized and it descends to a polarized endomorphism $h : Y \to Y$ of degree $q^{\dim Y}$ with $\pi \circ g_r = h \circ \pi$.*

The claim in the abstract about the building blocks of polarized endomorphisms, is justified by the remark below.

Remark 1.5.

(1) The Y in Theorem 1.4 is $\mathbb{Q}$-factorial and has at worst log terminal singularities.

(2) Suppose that the X in Theorem 1.4 is rationally connected. Then Y is also rationally connected. Suppose further that X has at worst terminal singularities and $(\dim X, \dim Y) = (3, 2)$. Then Y has at worst Du Val singularities by [8, Theorem 1.2.7]. So there is a composition $Y \to \hat{Y}$ of divisorial contractions and an extremal contraction $\hat{Y} \to B$ such that either $\dim B = 0$ and $\hat{Y}$ is a Du Val del Pezzo surface of Picard number 1, or $\dim B = 1$ and $\hat{Y} \to B \cong \mathbb{P}^1$ is a $\mathbb{P}^1$-fibration with all fibres irreducible. After replacing f by its power, h descends to polarized endomorphisms $\hat{h} : \hat{Y} \to \hat{Y}$, and $k : B \to B$ (of degree $q^{\dim B}$); see Theorems 1.6.

(3) By [2, Theorem 5.1], there are dense subsets $Y_0 \subset Y$ (for the Y in Theorem 1.4) and $B_0 \subset B$ (when $\dim B = 1$) such that for every $y \in Y_0$ (resp. $b \in B_0$) and for some $r(y) > 0$ (resp. $r(b) > 0$), $g^{r(y)}|W_y$ (resp. $\hat{h}^{r(b)}|\hat{Y}_b$) is a well-defined polarized endomorphism of the Fano fibre.

We remark that Noboru Nakayama has produced many examples of polarized f on abelian surfaces which are not scalar. The result below shows that this happens only on abelian surfaces and their quotients.

Theorem 1.6. *Let X be a normal projective surface. Suppose that $f : X \to X$ is an endomorphism such that $f^*P \equiv qP$ for some $q > 1$ and some big Weil $\mathbb{Q}$-divisor P. Then we have:*

(1) *f is polarized of degree q^2.*
(2) *There is an $s > 0$ such that $(f^s)^*|\mathrm{Weil}(X) = q^s \,\mathrm{id}$ unless X is Q-abelian with $\mathrm{rank}\mathrm{Weil}(X) \in \{3, 4\}$.*

More generally, we prove the two theorems below. Theorem 1.7 below includes Theorem 1.4 as a special case.

Theorem 1.7. *Let X be a $\mathbb{Q}$-factorial n-fold, with $n \in \{3, 4\}$, having only log terminal singularities and a polarized endomorphism f of degree $q^n > 1$. Let $X = X_0 \dashrightarrow X_1 \cdots \dashrightarrow X_r$ be a composition of divisorial contractions and flips. Replacing f by its positive power,* (I) *and* (II) *hold:*

(I) *The dominant rational maps $g_i : X_i \dashrightarrow X_i$ $(0 \le i \le r)$ (with $g_0 = f$) induced from f, are all holomorphic. Further, g_i^{-1} preserves* *each irreducible component of the exceptional locus of* $X_i \to X_{i+1}$ *(when it is divisorial) or of the flipping contraction* $X_i \to Z_i$ *(when* $X_i \dashrightarrow X_{i+1} = X_i^+$ *is a flip).*

(II) *Let* $\pi : W = X_r \to Y$ *be the contraction of a* K_W*-negative extremal ray* $\mathbb{R}_{\geq 0}[C]$*, with* $\dim Y \leq n-1$*. Then* $g := g_r$ *descends to a surjective endomorphism* $h : Y \to Y$ *of degree* $q^{\dim Y}$ *such that*

$$\pi \circ g = h \circ \pi.$$

For all $0 \leq i \leq r$*, all eigenvalues of* $g_i^*|N^1(X_i)$ *and* $h^*|N^1(Y)$ *are of modulus* q*; there are big line bundles* H_{X_i} *and* H_Y *satisfying*

$$g_i^* H_{X_i} \sim qH_{X_i}, \quad h^* H_Y \sim qH_Y.$$

Suppose further that either $\dim Y \leq 2$ *or* $\rho(Y) = 1$*. Then* H_W *and* H_Y *can be chosen to be ample and* g *and* h *are polarized.*

The contraction π below exists by the MMP for threefolds.

Theorem 1.8. *Let* X *be a* $\mathbb{Q}$*-factorial rationally connected threefold having at worst terminal singularities and a polarized endomorphism of degree* $>$ *1. Let* $X \dashrightarrow W$ *be a composition of divisorial contractions and flips, and* $\pi : W \to Y$ *an extremal contraction of non-birational type. Suppose either* $\dim Y \geq 1$*, or* $\dim Y = 0$ *and* W *is smooth. Then* X *is rational.*

The difficulty 1.9. In Theorem 1.4, if $X \to X_1$ is a divisorial contraction, one can descend a polarized endomorphism f on X to an one on X_1, but the latter may not be polarized any more because the pushfoward of a nef divisor may not be nef in dimension ≥ 3 (the first difficulty). If $X \dashrightarrow X_1$ is a flip, then in order to descend f on X to some holomorphic f_1 on X_1, one has to show that a power of f preserves the centre of the flipping contraction (the second difficulty). The second difficulty is taken care by a key lemma where the polarizedness is essentially used.

The question below is the generalization of Theorem 1.2 and the famous conjecture: every smooth Fano n-fold of *Picard number one* with a non-isomorphic surjective endomorophism, is $\mathbb{P}^n$ (for its affirmative solution when $n = 3$, see Amerik-Rovinsky-Van de Ven [1] and Hwang-Mok [4]).

Question 1.10. *Let* X *be a smooth Fano* n*-fold with a non-isomorphic polarized endomorphism. Is* X *rational ?*

For the recent development on endomorphisms of algebraic varieties, we refer to Amerik-Rovinsky-Van de Ven [1], Fujimoto-Nakayama [3], Hwang-Mok[4], S. -W. Zhang [16], as well as [10], [13].

Acknowledgement

I would like to thank Professor T. Katsura and organizers of the Kinosaki Annual Algebraic Geometry Symposium for the support and kind invitation, and Professor S. Mori for illustrating an example of a threefold conic bundle with an isolated point in the discriminant locus.

REFERENCES

[1] E. Amerik, M. Rovinsky and A. Van de Ven, A boundedness theorem for morphisms between threefolds, Ann. Inst. Fourier (Grenoble) **49** (2) (1999) 405–415.

[2] N. Fakhruddin, Questions on self maps of algebraic varieties, J. Ramanujan Math. Soc. **18** (2) (2003) 109–122.

[3] Y. Fujimoto and N. Nakayama, Complex projective manifolds which admit non-isomorphic surjective endomorphisms, RIMS Kokyuroku Bessatsu, to appear.

[4] J.-M. Hwang and N. Mok, Finite morphisms onto Fano manifolds of Picard number 1 which have rational curves with trivial normal bundles, J. Alg. Geom. **12** (2003) 627–651.

[5] V. A. Iskovskikh, On the rationality problem for algebraic threefolds, Proc. Steklov Inst. Math. **218** (3) (1997) 186–227.

[6] M. Miyanishi, Algebraic methods in the theory of algebraic threefolds— surrounding the works of Iskovskikh, Mori and Sarkisov, Algebraic varieties and analytic varieties (Tokyo, 1981), 69–99, Adv. Stud. Pure Math., **1**, North-Holland, Amsterdam, 1983.

[7] S. Mori and S. Mukai, Classification of Fano 3-folds with $B_2 \geq 2$, Manuscripta Math. **36** (1981/82), no. 2, 147–162; Erratum, Manuscripta Math. **110** (2003), no. 3, 407.

[8] S. Mori and Y. Prokhorov, On $\mathbb{Q}$-conic bundles, arXiv:math/**0603736**.

[9] N. Nakayama, On complex normal projective surfaces admitting non-isomorphic surjective endomorphisms, preprint September 2008.

[10] N. Nakayama and D.-Q. Zhang, Building blocks of étale endomorphisms of complex projective manifolds, preprint RIMS-**1577**, Kyoto Univ., 2007.

[11] N. Nakayama and D.-Q. Zhang, Polarized endomorphisms of complex normal varieties, preprint RIMS-**1613**, Kyoto Univ., 2007.

[12] J. -P. Serre, Analogues kählériens de certaines conjectures de Weil, Ann. of Math. **71** (1960) 392–394.

[13] D. -Q. Zhang, Dynamics of automorphisms of compact complex manifolds, Proceedings of The Fourth International Congress of Chinese Mathematicians (ICCM2007), Vol. **II**, pp. 678 - 689; also at: arXiv:**0801.0843**.

[14] D. -Q. Zhang, Polarized endomorphisms of uniruled varieties, preprint 2008.

[15] D. -Q. Zhang, Rationality of rationally connected varieties, preprint 2008.

[16] S.-W. Zhang, Distributions in algebraic dynamics, *Surveys in Differential Geometry*, Vol. **10**, pp. 381–430, Int. Press, 2006.

DEPARTMENT OF MATHEMATICS
NATIONAL UNIVERSITY OF SINGAPORE, 2 SCIENCE DRIVE 2, SINGAPORE 117543
E-mail address: matzdq@nus.edu.sg